Curve Generation via Dynamically Modulated Gravitational Fields

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Abstract

Infinitely differentiable (C^∞) curves are crucial in computer graphics, physical simulations, and geometric design. While traditional methods like splines and Bézier curves are effective, they face challenges regarding true infinite differentiability and flexible control. This paper proposes a novel physics-inspired framework for generating C^∞ smooth curves by continuously deforming a base curve (initially a straight line segment) under the influence of dynamically modulated gravitational fields. The deformation is driven by multiple gravitational control points, each defined by an adjustable magnitude and location. We explore four types of gravitational kernels governing the influence: Inverse-Power Law, Gaussian, Logarithmic Potential, and Hybrid combinations (Gaussian+Inverse-Power, Logarithmic+Gaussian). Regularization techniques are presented for kernels with singularities (Inverse-Power, Logarithmic) to ensure smooth behavior. The resulting curve is defined by the superposition of forces from all control points. A proof is provided demonstrating that the smoothness of the base curve and the (regularized) kernels guarantees C^∞ continuity of the final curve. This approach offers intuitive control over curve shape via control point parameters and kernel selection, with potential applications in graphics, simulation, and design requiring high degrees of smoothness and complex deformations.

1 Introduction

Infinitely differentiable, or C^∞, curves are of paramount importance in a diverse range of practical fields, each leveraging their inherent smoothness and continuity for critical applications. In computer graphics, these curves are essential for creating visually appealing and mathematically well-behaved shapes. Their smooth transitions and continuous changes in direction are fundamental for tasks such as motion planning and trajectory optimization in animation and virtual environments. For instance, Bézier curves, widely employed in computer graphics, serve as a cornerstone for designing smooth object outlines, creating intricate font designs, and defining animation paths [42]. The ability of these curves to represent complex shapes with a realistic appearance, surpassing the limitations of simple line segments, underscores their significance in the field. Furthermore, vector graphics systems utilize splines to represent curves that maintain their smoothness regardless of the viewing scale, highlighting the practical necessity of infinitely differentiable curves in digital visual media.

The realm of physical simulations also heavily relies on the properties of smooth curves. Differentiable simulators, increasingly utilized in computational physics, robotics, and machine learning, benefit significantly from smooth mathematical representations. These representations facilitate the efficient computation of gradients of physical processes, which is crucial for optimization and control tasks [1]. The use of Bézier curves to simulate natural movements, such as the subtle and continuous motion of a mouse cursor, exemplifies the application of smooth curves in creating realistic and physically plausible simulations [5].

In geometric design, the mathematical study of smooth shapes and spaces, known as differential geometry, highlights the fundamental role of smooth curves [7]. Applications range from the design of roads, where spline curves ensure gradual and safe transitions in curvature, to the aesthetic considerations in architectural and automotive design, where high levels of geometric continuity, closely related to infinite differentiability, are essential for visual appeal and functional performance.

Despite the effectiveness of traditional methods for smooth curve generation, such as splines and Bézier curves, they encounter certain challenges, particularly concerning the attainment of true infinite differentiability and the flexibility required for complex deformations. While splines offer versatility through their piecewise polynomial nature, achieving C^∞ continuity at the knot points, where polynomial segments meet, is not a standard feature and often requires specialized constructions. Standard polynomial splines typically provide C^k continuity, where k is related to the degree of the polynomials used. Increasing the degree to achieve higher-order continuity can introduce undesirable oscillations, such as Runge's phenomenon, and may not guarantee infinite differentiability. Moreover, imposing specific derivative conditions at the knots can sometimes compromise the overall smoothness of the curve. Although specialized types like CINAPACT-splines can achieve C^∞ smoothness, they are specific constructions rather than a general property of all spline types.

Bézier curves, while inherently smooth (C^∞) within their parametric domain, exhibit global control, meaning that modifying any single control point affects the entire curve shape [42]. This global influence can be a limitation when localized shape adjustments are desired. While increasing the number of control points can enable the creation of more complex shapes, it also increases computational costs and can potentially lead to numerical instability. Ensuring smooth connections between multiple Bézier curve segments requires careful and often non-intuitive placement of control points to achieve the desired levels of geometric and parametric continuity.

Inspired by the principles of physics, this paper proposes a novel approach to generating infinitely differentiable smooth curves through the concept of dynamically modulated gravitational fields. This framework draws upon the analogy of attractive forces, similar to gravity, to continuously deform a base curve. By introducing multiple gravitational control points, each characterized by an adjustable magnitude and location, the method aims to provide intuitive and flexible control over the curve's shape, enabling both localized and global deformations. The magnitude of a control point would correspond to the "strength" of its gravitational influence, while its location would determine the spatial extent of its effect.

2 Problem Formulation

The foundation of our framework begins with a simple geometric entity: a straight line segment defined by two fixed endpoints in a 2D plane. Let these endpoints be denoted as $\mathbf{P}_0 = (x_0, y_0)$ and $\mathbf{P}_1 = (x_1, y_1)$. This choice of a straight line as the base curve simplifies the initial mathematical formulation; however, it is important to note that the principles of this framework could be extended to other base parametric curves as well. The straight line segment can be represented parametrically by the equation:

$$\vec{C}(t) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1, \quad \text{for } t \in [0,1] \tag{1}$$

Expanding this equation into its component forms gives:

$$x(t) = (1-t)x_0 + tx_1 \tag{2}$$

$$y(t) = (1-t)y_0 + ty_1 \tag{3}$$

where t is a parameter that varies continuously from 0 to 1. Each value of t within this interval corresponds to a unique point along the straight line segment connecting $\mathbf{P}_0$ and $\mathbf{P}_1$. The core idea of our approach is to continuously deform this base curve under the influence of N gravitational control points. Each control point, denoted by G_i for $i = 1, 2, \ldots, N$, is defined by its magnitude M_i and its location $\mathbf{L}_i = (x_i, y_i)$ in the 2D plane. The magnitude M_i quantifies the strength of the gravitational influence exerted by the i-th control point, while its location $\mathbf{L}_i$ specifies where this influence is centered.

The deformation of the base curve is achieved through the cumulative effect of the gravitational fields generated by these control points. A control point with a larger magnitude will exert a stronger "gravitational pull," resulting in a greater displacement of the curve in its vicinity. Conversely, a control point with a smaller magnitude will have a weaker influence. The distance between a control point G_i located at $\mathbf{L}_i$ and a point $\vec{C}(t)$ on the curve is a critical factor in determining the strength and nature of the deformation. Control points positioned close to the curve will exert a more localized influence, potentially causing sharper bends or more pronounced deviations in that specific region. Control points situated farther away will contribute to broader, smoother deformations of the overall curve shape. The precise manner in which this gravitational influence decays with distance is governed by the mathematical formulation of the chosen gravitational kernel.

The process of deforming the curve can be envisioned as a continuous displacement of each point $\vec{C}(t)$ on the initial straight line segment. This displacement, denoted by $\Delta\vec{C}(t)$, is the result of the net gravitational force acting on that point due to all N control points. For each control point G_i, the force exerted on $\vec{C}(t)$ will depend on its magnitude M_i, its location $\mathbf{L}_i$, the distance $r_{it} = \left\|\mathbf{L}_i - \vec{C}(t)\right\|$, and the specific form of the gravitational kernel being employed. The total displacement $\Delta\vec{C}(t)$ at a point $\vec{C}(t)$ is then calculated as the vector sum of the individual forces (or influences) from all N control points. The final, deformed curve $\vec{C'}(t)$ is obtained by adding this displacement vector to the original position vector of the point on the base curve:

$$\vec{C'}(t) = \vec{C}(t) + \Delta\vec{C}(t) \tag{4}$$

This formulation embodies the principle of superposition, where the total effect at any point on the curve is the sum of the individual contributions from each control point. The magnitude and location of these control points serve as intuitive parameters for manipulating the curve's shape, while the selection of the gravitational kernel dictates the spatial characteristics of their influence.

3 Gravitational Field Models

The heart of our curve generation framework lies in the mathematical models used to represent the gravitational influence of each control point. These models, known as gravitational kernels, define how the "gravitational pull" of a control point varies with distance and thus determines the shape of the resulting deformation. We propose the use of four distinct types of kernels: the Inverse-Power Law Kernel, the Gaussian Kernel, the Logarithmic Potential Kernel, and Hybrid Kernels that combine the features of these fundamental kernels.

3.1 Inverse-Power Law Kernel

The Inverse-Power Law Kernel models a force field where the strength of the influence decays with the distance r from the source as an inverse power of that distance [12]. The gravitational potential $\Phi(r)$ generated by a control point of magnitude M at a distance r is mathematically described as:

$$\Phi(r) = -\frac{M}{r^\alpha} \tag{5}$$

where $\alpha > 0$ is the exponent that governs the rate of decay of the potential with distance. A larger value of α indicates a more rapid decay. The gravitational force $\vec{F}(r)$, which is responsible for deforming the curve, is the negative gradient of the potential:

$$\vec{F}(r) = -\nabla\Phi(r) = -\frac{d\Phi}{dr}\hat{r} = -\frac{\alpha M}{r^{\alpha+1}}\hat{r} \tag{6}$$

Here, $\hat{r}$ is the unit vector pointing from the point on the curve towards the control point. A significant challenge associated with the Inverse-Power Law kernel is the singularity that occurs at $r = 0$, where both the potential and the force tend to infinity. This singularity can lead to undesirable non-smooth deformations if not handled appropriately. To address this, a common regularization technique involves modifying the kernel at small distances by introducing a small positive constant $\epsilon^2 > 0$ [15]. A typical regularization replaces r^2 with $r^2 + \epsilon^2$ in the denominator (or equivalent for r^α). Applying this to the potential (using $(r^2 + \epsilon^2)^{\alpha/2}$ as the denominator base) gives the regularized form:

$$\text{Regularized Potential: } \Phi_\epsilon(r) = -\frac{M}{(r^2 + \epsilon^2)^{\alpha/2}} \tag{7}$$

The corresponding regularized force is then:

$$\text{Regularized Force: } \vec{F}_\epsilon(r) = -\nabla\Phi_\epsilon(r) = -\frac{d\Phi_\epsilon}{dr}\hat{r} = -\frac{\alpha M r}{(r^2 + \epsilon^2)^{(\alpha+2)/2}}\hat{r} \tag{8}$$

This regularization ensures that as $r \to 0$, the potential approaches a finite value of $-M/\epsilon^\alpha$, and the force approaches 0, effectively smoothing out the singularity and ensuring a well-behaved kernel near the control point. The parameter ϵ controls the scale at which this smoothing occurs. The Inverse-Power Law kernel is advantageous for its long-range influence, making it suitable for global deformations of the curve [12]. The exponent α provides a means to tune the strength and reach of the gravitational effect. However, the choice of the regularization parameter ϵ is crucial to balance the smoothing of the singularity with preserving the desired long-range behavior.

3.2 Gaussian Kernel

The Gaussian Kernel is employed to model localized, fine-scale deformations due to its characteristic rapid and smooth decay of influence with distance [19]. The mathematical formulation for the gravitational potential $\Phi(r)$ generated by a control point of magnitude M using a Gaussian kernel is given by:

$$\Phi(r) = -Me^{-\frac{r^2}{2\sigma^2}} \tag{9}$$

where $\sigma > 0$ represents the standard deviation of the Gaussian function. This parameter plays a critical role in determining the spatial spread of the kernel's influence [6]. A smaller value of σ results in a more concentrated, highly localized deformation effect, while a larger σ leads to a broader, more gradual influence [6]. The gravitational force $\vec{F}(r)$ derived from this potential is:

$$\vec{F}(r) = -\nabla\Phi(r) = -\frac{d\Phi}{dr}\hat{r} = -\frac{Mr}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\hat{r} \tag{10}$$

An important property of the Gaussian kernel is that it, along with all its derivatives, is infinitely differentiable (C^∞) for all $r \geq 0$ [10]. Furthermore, the force derived from the Gaussian potential is zero at $r = 0$, thus inherently avoiding any singularity at the location of the control point. The primary advantage of the Gaussian kernel lies in its ability to produce highly localized and smooth deformations, making it particularly useful for introducing fine details or making precise adjustments to specific regions of the curve [19]. A limitation of the Gaussian kernel is its relatively short range of significant influence compared to kernels like the Inverse-Power Law or Logarithmic Potential [6]. The choice of the standard deviation σ is therefore crucial to control the scale of the localized deformations.

3.3 Logarithmic Potential Kernel

The Logarithmic Potential Kernel models a gravitational influence that exhibits a long-range decay, although its decay rate at large distances is generally slower than that of the Inverse-Power Law kernel [25]. In two dimensions, the gravitational potential $\Phi(r)$ for a control point of magnitude M is defined as:

$$\Phi(r) = M\ln(r) \tag{11}$$

where r is the distance from the control point. The corresponding gravitational force $\vec{F}(r)$ is given by the gradient of the potential (note: typically force is $-\nabla\Phi$, but potential wells are often defined with opposite sign conventions; here we follow the text's implication of force being $\nabla\Phi$ for this kernel, assuming an attractive force is desired, the potential sign might need adjustment later):

$$\vec{F}(r) = \nabla\Phi(r) = \frac{d\Phi}{dr}\hat{r} = \frac{M}{r}\hat{r} \tag{12}$$

Similar to the Inverse-Power Law kernel, the Logarithmic Potential kernel has a singularity at $r = 0$, where $\ln(r) \to -\infty$ and the force tends to infinity [25]. Regularization is therefore necessary to ensure the kernel is well-behaved near the control point [27]. A common regularization approach involves replacing r with $\sqrt{r^2 + \epsilon^2}$, where $\epsilon > 0$ is a small positive constant. This yields the regularized potential:

$$\text{Regularized Potential: } \Phi_\epsilon(r) = M\ln(\sqrt{r^2 + \epsilon^2}) = \frac{1}{2}M\ln(r^2 + \epsilon^2) \tag{13}$$

The corresponding regularized force becomes:

$$\text{Regularized Force: } \vec{F}_\epsilon(r) = \nabla\Phi_\epsilon(r) = \frac{d\Phi_\epsilon}{dr}\hat{r} = \frac{Mr}{r^2 + \epsilon^2}\hat{r} \tag{14}$$

With this regularization, as $r \to 0$, the potential approaches $M \ln(\epsilon)$, which is finite, and the force approaches 0. The Logarithmic Potential kernel offers a distinct profile of long-range influence, which can be advantageous for achieving global deformations with different characteristics compared to the Inverse-Power Law [25]. However, its influence at very large distances can be weaker, and the singularity at $r = 0$ necessitates careful regularization.

3.4 Hybrid Kernels

To leverage the strengths of the individual kernels and achieve more complex and nuanced deformation effects, we propose the use of hybrid kernels. These are constructed by taking a weighted linear combination of the potentials of two or more base kernels [31]. The weights allow for controlling the relative contribution of each base kernel to the overall deformation field.

3.4.1 Gaussian + Inverse-Power Hybrid

This hybrid kernel combines the localized, fine-scale control provided by the Gaussian kernel with the long-range influence of the Inverse-Power Law kernel [35]. The hybrid potential $\Phi_{hybrid}(r)$ is defined as:

$$\Phi_{hybrid}(r) = w_1 \Phi_{\text{Gaussian}}(r) + w_2 \Phi_{\text{InversePower},\epsilon_2}(r) \tag{15}$$

$$= -w_1 M_1 e^{-\frac{r^2}{2\sigma_1^2}} - w_2 \frac{M_2}{(r^2 + \epsilon_2^2)^{\alpha_2/2}} \tag{16}$$

where $w_1, w_2 \geq 0$ are the weights such that $w_1 + w_2 = 1$, and M_1, σ_1 are the magnitude and standard deviation of the Gaussian kernel, while $M_2, \alpha_2, \epsilon_2$ are the magnitude, exponent, and regularization parameter of the Inverse-Power Law kernel, respectively. The resulting hybrid force $\vec{F}_{hybrid}(r)$ is the negative gradient of this potential:

$$\vec{F}_{hybrid}(r) = \left(-w_1 \frac{M_1 r}{\sigma_1^2} e^{-\frac{r^2}{2\sigma_1^2}} - w_2 \frac{\alpha_2 M_2 r}{(r^2 + \epsilon_2^2)^{(\alpha_2+2)/2}} \right) \hat{r} \tag{17}$$

This combination allows for simultaneous control over both local details and the global shape of the curve. The weights w_1 and w_2 determine the relative importance of the localized and long-range effects.

3.4.2 Logarithmic + Gaussian Hybrid

This hybrid kernel combines the long-range influence of the Logarithmic Potential kernel with the localized influence of the Gaussian kernel (inspired by [39], [31]). The hybrid potential $\Phi_{hybrid}(r)$ is given by combining the regularized Logarithmic and Gaussian potentials (assuming force for Log is $\nabla\Phi$ and for Gaussian is $-\nabla\Phi$, we combine potentials accordingly, or assume a consistent force direction, e.g., attractive, meaning $\vec{F} = -k\hat{r}$ where $k > 0$. Let's assume both are attractive potentials, so force is $-\nabla\Phi$ for both, adjusting the sign of the Log potential term):

$$\Phi_{hybrid}(r) = w_1 \Phi_{\text{Logarithmic},\epsilon_1}(r) + w_2 \Phi_{\text{Gaussian}}(r) \tag{18}$$

$$= -\frac{1}{2} w_1 M_1 \ln(r^2 + \epsilon_1^2) - w_2 M_2 e^{-\frac{r^2}{2\sigma_2^2}} \tag{19}$$

with weights $w_1, w_2 \geq 0$ ($w_1 + w_2 = 1$) and parameters M_1, ϵ_1 for the regularized Logarithmic kernel and M_2, σ_2 for the Gaussian kernel. Note the negative sign added to the Logarithmic term to ensure consistency with an attractive force derived via $\vec{F} = -\nabla\Phi$. The hybrid force $\vec{F}_{hybrid}(r)$ is the negative gradient of this potential:

$$\vec{F}_{hybrid}(r) = \left(w_1 \frac{M_1 r}{r^2 + \epsilon_1^2} - w_2 \frac{M_2 r}{\sigma_2^2} e^{-\frac{r^2}{2\sigma_2^2}} \right) \hat{r} \tag{20}$$

This hybrid offers an alternative approach to achieving both global and local influences on the curve's shape, potentially resulting in unique deformation characteristics due to the nature of the Logarithmic potential.

4 Continuous Deformation Formula

The continuous deformation of the base straight line segment $\vec{C}(t)$ under the influence of N gravitational control points is achieved by calculating the displacement $\Delta\vec{C}(t)$ at each point $\vec{C}(t)$ due to the combined gravitational forces exerted by all control points. The deformed curve $\vec{C}'(t)$ is then given by:

$$\vec{C}'(t) = \vec{C}(t) + \Delta\vec{C}(t) \tag{21}$$

The total displacement vector $\Delta\vec{C}(t)$ is the vector sum of the individual forces contributed by each control point G_i located at $\mathbf{L}_i$ with magnitude M_i and employing a specific kernel. For each control point, the force exerted on $\vec{C}(t)$ depends on the distance $r_{it} = \left\|\mathbf{L}_i - \vec{C}(t)\right\|$ and the unit vector $\hat{r}_{it} = \frac{\mathbf{L}_i - \vec{C}(t)}{\left\|\mathbf{L}_i - \vec{C}(t)\right\|}$ pointing from $\vec{C}(t)$ to $\mathbf{L}_i$. The total displacement is thus:

$$\Delta\vec{C}(t) = \sum_{i=1}^{N} \vec{F}_i(r_{it}) \tag{22}$$

where $\vec{F}_i(r_{it})$ is the force vector exerted by the i-th control point at a distance r_{it}. This force vector is given by $F_i(r_{it})\hat{r}_{it}$ (or potentially $-F_i(r_{it})\hat{r}_{it}$ depending on attractive/repulsive convention and potential definition, but consistent with the forces derived above, it represents the vector force). The magnitude M_i scales the strength of this force. The location $\mathbf{L}_i$ determines r_{it} and $\hat{r}_{it}$, influencing the deformation's locality. The principle of superposition ensures that the total deformation is the sum of individual contributions. Since the base curve $\vec{C}(t)$ is C^∞ and the force fields from the Gaussian kernel and the regularized Inverse-Power Law and Logarithmic kernels are also C^∞ (as shown in the next section), the resulting deformed curve $\vec{C}'(t)$ is infinitely differentiable.

5 Proof of Infinite Differentiability

5.1 Mathematical Preliminaries: Infinite Differentiability

A function $f : U \to \mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$, is said to be infinitely differentiable (or of class C^∞) if all its partial derivatives of all orders exist and are continuous on U. For a scalar function $\Phi(x, y)$ in two dimensions ($n = 2, m = 1$), this means that partial derivatives like $\frac{\partial^k \Phi}{\partial x^i \partial y^{k-i}}$ exist and are continuous for all non-negative integers k and $0 \le i \le k$. For a vector field $\vec{F}(x, y) = (F_x(x, y), F_y(x, y))$, infinite differentiability requires that both component functions F_x and F_y are infinitely differentiable.

In our context, the gravitational potential $\Phi(r)$ associated with a control point G_i at location $\mathbf{L}_i = (x_i, y_i)$ is primarily a function of the distance r. Since $r = \sqrt{(x - x_i)^2 + (y - y_i)^2}$, where (x, y) are the coordinates of a point $\vec{C}(t)$ on the curve, the potential Φ can be viewed as a function $\Phi(x, y)$. The gravitational force $\vec{F}(x, y)$ is related to the gradient of the potential (typically $\vec{F} = -\nabla\Phi$ for attractive forces from potentials like Gaussian and Inverse-Power, or $\vec{F} = \nabla\Phi$ as defined for the Logarithmic kernel, though we adjusted the Log potential in Eq. 19 for consistency). The force $\vec{F}(x, y)$ is a vector field whose components are partial derivatives of Φ with respect to x and y:

$$F_x = -\frac{\partial \Phi}{\partial x} \tag{23}$$

$$F_y = -\frac{\partial \Phi}{\partial y} \tag{24}$$

(assuming $\vec{F} = -\nabla\Phi$).

The final deformed curve is $\vec{C}'(t) = \vec{C}(t) + \Delta\vec{C}(t)$. Since $\vec{C}(t)$ is a linear function of t, it is C^∞. The displacement $\Delta\vec{C}(t)$ (Eq. 22) is a sum of force vectors $\vec{F}_i(r_{it})$. Each force $\vec{F}_i$ depends on the position $\vec{C}(t) = (x(t), y(t))$. Thus, $\Delta\vec{C}(t)$ is a composition of functions: the kernel function (which depends on x, y) and the base curve functions $x(t), y(t)$. The composition of C^∞ functions is C^∞. Therefore, if we show that each force kernel $\vec{F}_i(x, y)$ is C^∞ with respect to (x, y), then $\Delta\vec{C}(t)$ will be C^∞ with respect to t, and consequently, $\vec{C}'(t)$ will be C^∞.

5.2 Proofs for Individual Kernels

5.2.1 Regularized Inverse-Power Law Kernel

The regularized potential is given by:

$$\Phi_\epsilon(r) = -\frac{M}{(r^2+\epsilon^2)^{\alpha/2}} \tag{25}$$

where $r^2 = (x-x_i)^2 + (y-y_i)^2$ and $\epsilon > 0$. Let $u(x,y) = (x-x_i)^2 + (y-y_i)^2 + \epsilon^2$. This is a polynomial in x and y. Since polynomials are infinitely differentiable, $u(x,y)$ is C^∞. Furthermore, since $(x-x_i)^2 \geq 0$ and $(y-y_i)^2 \geq 0$, and $\epsilon^2 > 0$, we have $u(x,y) \geq \epsilon^2 > 0$ for all (x,y). The potential can be written as $\Phi_\epsilon(x,y) = -M(u(x,y))^{-\alpha/2}$. Consider the function $v(z) = z^{-\alpha/2}$. Since $\alpha > 0$, this function is infinitely differentiable for $z > 0$. Since $u(x,y) > 0$ and is C^∞, and $v(z)$ is C^∞ for $z > 0$, the composition $\Phi_\epsilon(x,y) = -Mv(u(x,y))$ is infinitely differentiable by the chain rule. The force components are:

$$F_{\epsilon,x} = -\frac{\partial \Phi_\epsilon}{\partial x} = -\frac{\alpha M(x-x_i)}{(r^2+\epsilon^2)^{(\alpha+2)/2}} \tag{26}$$

$$F_{\epsilon,y} = -\frac{\partial \Phi_\epsilon}{\partial y} = -\frac{\alpha M(y-y_i)}{(r^2+\epsilon^2)^{(\alpha+2)/2}} \tag{27}$$

The denominator $(r^2+\epsilon^2)^{(\alpha+2)/2} = (u(x,y))^{(\alpha+2)/2}$ is an infinitely differentiable function (composition of $u(x,y)$ and $z^{(\alpha+2)/2}$) which is never zero. The numerators $\alpha M(x-x_i)$ and $\alpha M(y-y_i)$ are polynomials in x and y, hence C^∞. Since $F_{\epsilon,x}$ and $F_{\epsilon,y}$ are ratios of infinitely differentiable functions where the denominator is never zero, they are infinitely differentiable. Thus, the force field $\vec{F}_\epsilon(x,y)$ is C^∞.

5.2.2 Gaussian Kernel

The potential is:

$$\Phi(r) = -Me^{-\frac{r^2}{2\sigma^2}} \tag{28}$$

where $r^2 = (x-x_i)^2 + (y-y_i)^2$ and $\sigma > 0$. Let $w(x,y) = -\frac{r^2}{2\sigma^2} = -\frac{(x-x_i)^2+(y-y_i)^2}{2\sigma^2}$. This is a polynomial in x and y, hence infinitely differentiable. The potential is $\Phi(x,y) = -Me^{w(x,y)}$. The exponential function e^z is infinitely differentiable for all $z \in \mathbb{R}$. By the chain rule, the composition $\Phi(x,y) = -Me^{w(x,y)}$ is infinitely differentiable. The force components are:

$$F_x = -\frac{\partial \Phi}{\partial x} = -\frac{M(x-x_i)}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}} \tag{29}$$

$$F_y = -\frac{\partial \Phi}{\partial y} = -\frac{M(y-y_i)}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}} \tag{30}$$

These components are products of polynomials (e.g., $-\frac{M}{\sigma^2}(x-x_i)$) and the infinitely differentiable function $e^{w(x,y)}$. The product of infinitely differentiable functions is infinitely differentiable. Thus, F_x and F_y are C^∞, and the force field $\vec{F}(x,y)$ is C^∞.

5.2.3 Regularized Logarithmic Potential Kernel

The regularized potential is:

$$\Phi_\epsilon(r) = \frac{1}{2}M\ln(r^2+\epsilon^2) \tag{31}$$

where $r^2 = (x-x_i)^2+(y-y_i)^2$ and $\epsilon > 0$. (Note: We use the form from Eq. 19 for consistency, assuming $\vec{F} = -\nabla\Phi$). Let's redefine $\Phi_\epsilon(r) = -\frac{1}{2}M\ln(r^2+\epsilon^2)$ for an attractive force. Let $u(x,y) = (x-x_i)^2+(y-y_i)^2+\epsilon^2$. As established before, $u(x,y)$ is C^∞ and $u(x,y) \geq \epsilon^2 > 0$. The potential is $\Phi_\epsilon(x,y) = -\frac{1}{2}M\ln(u(x,y))$. The natural logarithm function $\ln(z)$ is infinitely differentiable for $z > 0$. By the chain rule, the composition

$\Phi_\epsilon(x,y)$ is infinitely differentiable. The force components are:

$$F_{\epsilon,x} = -\frac{\partial \Phi_\epsilon}{\partial x} = \frac{M(x-x_i)}{r^2+\epsilon^2} \tag{32}$$

$$F_{\epsilon,y} = -\frac{\partial \Phi_\epsilon}{\partial y} = \frac{M(y-y_i)}{r^2+\epsilon^2} \tag{33}$$

The denominator $r^2+\epsilon^2 = u(x,y)$ is C^∞ and never zero. The numerators $M(x-x_i)$ and $M(y-y_i)$ are polynomials, hence C^∞. Since $F_{\epsilon,x}$ and $F_{\epsilon,y}$ are ratios of infinitely differentiable functions with a non-zero denominator, they are infinitely differentiable. Thus, the force field $\vec{F}_\epsilon(x,y)$ is C^∞.

5.3 Proofs for Hybrid Kernels

5.3.1 Gaussian + Inverse-Power Hybrid Kernel

The hybrid potential is:

$$\Phi_{hybrid}(r) = w_1\Phi_{\text{Gaussian}}(r) + w_2\Phi_{\text{InversePower},\epsilon_2}(r) \tag{34}$$

where $w_1, w_2 \geq 0$ and $w_1+w_2 = 1$. We have already shown that $\Phi_{\text{Gaussian}}(r)$ (as $\Phi(x,y)$) and $\Phi_{\text{InversePower},\epsilon_2}(r)$ (as $\Phi_{\epsilon_2}(x,y)$) are infinitely differentiable functions of (x,y). A linear combination of infinitely differentiable functions is also infinitely differentiable. Therefore, $\Phi_{hybrid}(x,y)$ is infinitely differentiable. The force derived from this potential is:

$$\vec{F}_{hybrid}(r) = -\nabla\Phi_{hybrid}(r) \tag{35}$$

$$= w_1(-\nabla\Phi_{\text{Gaussian}}(r)) + w_2(-\nabla\Phi_{\text{InversePower},\epsilon_2}(r)) \tag{36}$$

$$= w_1\vec{F}_{\text{Gaussian}}(r) + w_2\vec{F}_{\text{InversePower},\epsilon_2}(r) \tag{37}$$

Since $\vec{F}_{\text{Gaussian}}(x,y)$ and $\vec{F}_{\text{InversePower},\epsilon_2}(x,y)$ are infinitely differentiable vector fields, their linear combination $\vec{F}_{hybrid}(x,y)$ is also an infinitely differentiable vector field.

5.3.2 Logarithmic + Gaussian Hybrid Kernel

The hybrid potential (using the form ensuring attractive force) is:

$$\Phi_{hybrid}(r) = w_1\Phi_{\text{Logarithmic},\epsilon_1}(r) + w_2\Phi_{\text{Gaussian}}(r) \tag{38}$$

where:

$$\Phi_{\text{Logarithmic},\epsilon_1}(r) = -\frac{1}{2}M_1\ln(r^2+\epsilon_1^2) \tag{39}$$

and $w_1, w_2 \geq 0$, $w_1+w_2 = 1$. We have shown that $\Phi_{\text{Logarithmic},\epsilon_1}(x,y)$ and $\Phi_{\text{Gaussian}}(x,y)$ are infinitely differentiable. Thus, their linear combination $\Phi_{hybrid}(x,y)$ is also infinitely differentiable. The force is:

$$\vec{F}_{hybrid}(r) = -\nabla\Phi_{hybrid}(r) \tag{40}$$

$$= w_1(-\nabla\Phi_{\text{Logarithmic},\epsilon_1}(r)) + w_2(-\nabla\Phi_{\text{Gaussian}}(r)) \tag{41}$$

$$= w_1\vec{F}_{\text{Logarithmic},\epsilon_1}(r) + w_2\vec{F}_{\text{Gaussian}}(r) \tag{42}$$

As $\vec{F}_{\text{Logarithmic},\epsilon_1}(x,y)$ and $\vec{F}_{\text{Gaussian}}(x,y)$ are infinitely differentiable vector fields, their linear combination $\vec{F}_{hybrid}(x,y)$ is also an infinitely differentiable vector field.

5.4 Conclusion of Proof

Since the base curve $\vec{C}(t) = (x(t), y(t))$ is C^∞ with respect to t, and each force kernel $\vec{F}_i(x,y)$ (whether basic or hybrid, with regularization where needed) is C^∞ with respect to (x,y), the composition $\vec{F}_i(\vec{C}(t))$ is C^∞ with respect to t. The total displacement is:

$$\Delta\vec{C}(t) = \sum_{i=1}^{N}\vec{F}_i(\vec{C}(t)) \tag{43}$$

This is a sum of C^∞ functions, and is therefore C^∞. Finally, the deformed curve is:

$$\vec{C}'(t) = \vec{C}(t) + \Delta\vec{C}(t) \tag{44}$$

This is the sum of two C^∞ functions, and is thus infinitely differentiable (C^∞) with respect to the parameter t.

6 Conclusion

This paper introduces a novel method for generating infinitely differentiable smooth curves by leveraging the concept of dynamically modulated gravitational fields. The framework begins with a straight line segment that is continuously deformed by the influence of N gravitational control points. Each control point is characterized by its magnitude and location, and its influence on the curve is governed by a chosen gravitational kernel. We have explored four types of kernels: the Inverse-Power Law, the Gaussian, the Logarithmic Potential, and hybrid kernels formed by combining the Gaussian with the Inverse-Power Law and the Logarithmic Potential. For kernels exhibiting singularities at the origin (Inverse-Power Law and Logarithmic), we derived regularized formulations to ensure finite and smooth behavior for all distances. A continuous deformation formula was presented, describing the displacement of each point on the base curve as the vector sum of the forces exerted by all control points. Crucially, we proved that the inherent smoothness of the Gaussian kernel and the smooth nature of the regularized Inverse-Power Law and Logarithmic kernels, combined with the smoothness of the base curve, guarantee that the resulting deformed curves are infinitely differentiable (C^∞). This offers a significant advantage over traditional piecewise polynomial methods like standard splines, where achieving C^∞ continuity can be challenging.

The proposed method holds promising potential for various applications. In computer graphics, it could be used to generate highly smooth and intricate curves for object design, animation paths, and special effects, with the localized control of the Gaussian kernel being particularly useful for adding fine details [42]. For physical simulations, this framework could provide a means to generate smooth and continuous trajectories for particles or deformable objects, potentially offering a more natural way to model interactions [1]. In geometric design, the ability to create curves with specific smoothness properties could be valuable in engineering applications such as airfoil design or road construction [7].

Future research directions could explore methods for automatically determining optimal control point configurations to achieve desired curve shapes. Investigating the use of other types of gravitational kernels or more complex hybrid combinations could further expand the range of achievable deformations. Extending this framework to generate infinitely differentiable smooth surfaces by deforming a base planar patch in 3D space using gravitational control points is another promising avenue. Analyzing the computational efficiency of this method and developing techniques for real-time curve generation and manipulation would be crucial for practical applications. Finally, a comparative study of this method against existing curve generation techniques, using quantitative metrics and user evaluations, would help to establish its strengths and limitations more definitively.

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